Teaching and Learning RESEARCH BRIEFING

Number 13

Fractions: difficult but crucial in mathematics learning

Quantities represented by natural numbers are easily understood. We can count and say how many oranges are in a bag. But fractions cause difficulty to most people because they involve relations between quantities. What is $\frac{1}{2}$? One half of what? If Ali and Jazmine both spent $\frac{1}{2}$ of their pocket money on snacks, they may not have spent the same amount of money each. We have developed a teaching programme which boosts pupils' understanding of the relative nature of fractions.

Most pupils in Years 4 and 5 have not grasped the relative nature of fractions as numbers. Their difficulty is primarily conceptual. →	Teaching pupils about fractions must include a focus on the logical relations involved in this concept.
Pupils have some intuitive understanding of the relative nature of fractions from their experiences with division. →	Teaching about logical relations should build on pupils' intuitions.
Teaching programmes that start from pupils' intuitions about sharing and establish connections to fractions as numbers can have a positive impact on pupils' learning. →	Teacher education should make teachers aware of pupils' intuitive understanding of the logic of fractions and the situations in which they are understood most easily.

The research

This Teaching and Learning Research Programme project addresses children's basic ideas about fractions. It is well documented that most children have little difficulty in understanding natural numbers, but that most are challenged by rational numbers – or fractions, as they are called in primary school. The relative nature of fractions is a source of difficulty for pupils. It requires that they realise that the same fraction may refer to different quantities ($^1/_2$ of 8 and $^1/_2$ of 12 are different) and that different fractions may be equivalent because they refer to the same quantity ($^1/_3$ and $^3/_9$, for example). It is not possible for pupils to make further progress in mathematics or to take advanced courses in secondary school without a sound grasp of the relative nature of rational numbers.

Research on fractions has shown that many of the mistakes which pupils make when working with fractions can be seen as a consequence of their failure to understand that natural and rational numbers involve different ideas. One well-documented error that pupils make with fractions is to think that, for example, $^1/_3$ of a cake is smaller than $^1/_5$ because 3 is less than 5. Yet most children readily recognise that a cake shared among three children gives bigger portions than the same cake shared among five children. Because children do show good insight into some aspects of fractions when they are thinking about division, mathematics educators have begun to investigate whether these situations could be used as a starting point for teaching fractions.

In order to see whether this could be a good approach for pupils in the UK, we carried out five studies: two surveys of children's performance in fraction problems, one detailed analysis of pupils' reasoning about fractions, and two teaching experiments.

Surveys of pupils' performance

Our aim was to see whether pupils who are just starting to learn about fractions in school have intuitions about fractions that could be used as a basis for their further learning. We presented them with problems in two types of practical situation: part–whole, and division.

In **part–whole situations,** which are typically used to introduce fractions in primary school, the denominator indicates the number of equal parts into which a whole was cut and the numerator indicates the number of parts taken: for example, if a chocolate bar was cut into 4 equal parts and Sarah ate 1, Sarah ate $^1/_4$ of the chocolate.

In **division situations**, the numerator refers to the number of items being shared and the denominator refers to number of recipients: if 1 chocolate is shared among 4 children, the number 1 refers to the number of chocolates being shared and the number 4 refers to the number of recipients; the fraction $^1/_4$ indicates both the division – 1 divided by 4 – and the portion that each one receives.

The directions offered by the Numeracy Strategy for the teaching of fractions have many examples of part–whole situations but do not include examples of sharing as situations in which the meaning of fractions can be explored by pupils.

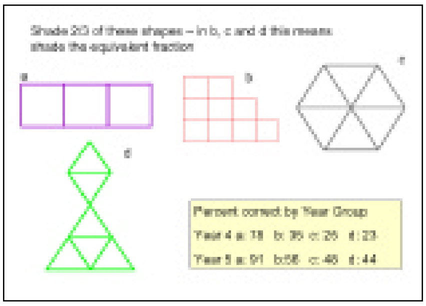

Figure 1

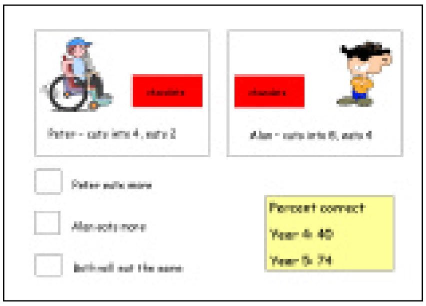

Figure 2

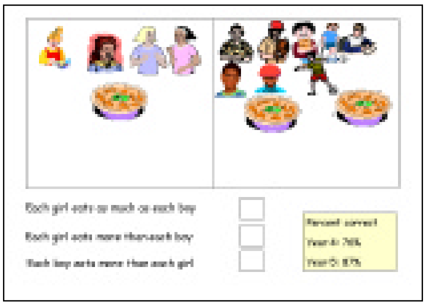

Figure 3

Two surveys were carried out in eight schools in Oxford and London (including two cohorts), covering a total of 449 children in Years 4 and 5. In both surveys they solved items about equivalent fractions in part–whole and division situations. In the first survey, the examples used for part–whole situations (taken from Hart et al., 1984) were typical of the exercises that the pupils would have encountered in school. For example, in one item they were asked to paint $^2/_3$ of two figures divided into 6 equal parts and of one figure divided into 9 equal parts (Figure 1). This involves thinking about equivalent fractions because they would actually have to paint $^4/_6$ and $^6/_9$ of the figures respectively. In the second survey, we designed items for the part–whole situations that were more comparable in format to those used in the division situation (Figures 2 and 3). Surprisingly, pupils were much better when solving the division problems, about which they had received no systematic teaching, than solving part–whole problems, used often in the classroom. The proportion of correct responses in both surveys was significantly higher in division problems than in part–whole problems; the effect size in both studies was large.

Detailed analyses of pupils' reasoning

Once we found that pupils were better at solving fractions problems about division than part–whole situations, it became important to understand how they achieved this success. We worked with small groups of pupils (four to six at a time) in Years 4 and 5 over a total of five sessions of about 45 minutes each and asked them to solve several problems, recording their strategies when comparing fractions and their explanations to each other during these sessions. In one problem, for example, they were told that six children went to a pizzeria and ordered two pizzas. They were asked to suggest different ways in which the pizzas could be shared fairly. After they had done so, they were asked whether the children would eat the same amount either way.

The children's arguments were often based on the logic of division. Essentially they argued that if the pizzas were shared fairly and completely, the way in which the pizzas were cut did not matter. For example, one child said: 'They're the same amount of people, the same amount of pizzas, and that means the same amount of fractions.' Another child argued: 'Because first we did one pizza and did it in six pieces, we found out that two sixths was one third, so if we used both of them then we could just use one third each.' But the children also argued that the number of pieces and the size of pieces compensated for each other: $^2/_6$ and $^1/_3$ are just 'a different way in fractions, and it [$^2/_6$] doubled [the number of pieces] to make it littler, and halving [the number of pieces] makes it bigger.

Children's drawings showed the use of the logic of division when the children established correspondences between the items that were being shared and the recipients, without displaying much concern with the partitioning of the pizzas (Figure 4a). Actually, a concern with partitioning and perceptual comparison often left the children unable to reach a conclusion because they had difficulty drawing the pizzas and doing the partitioning with sufficient precision (Figure 4b). The analysis of the children's strategies suggested that a focus on partitioning and perceptual comparison moved them away from thinking about the logic of division.

The teaching experiments

We carried out two experiments: one in which researchers taught the children in small groups outside the classroom and the other in which teachers in five different classrooms in four schools used sharing problems to introduce their children to fraction concepts. The schools varied considerably in intake. The taught group's performance was compared in a pre- and two post-tests to that of pupils in the same year groups and in the same school who had not participated in these teaching sessions and had received only their regular instruction on fractions. However, they participated in extra numeracy activities that included multiplication and division without including fractions and that were presented in ways that were similar to the tasks used for the teaching of fractions. Between five and eight teaching sessions were required for the children to go through all the problems. The taught groups did not differ from the comparison groups at pre-test. At both post-tests, one at the end of the teaching period and the second about eight weeks later, the taught groups significantly outperformed the comparison groups in the same fractions test. The effect size was small but sustained over the two-month period to the delayed post-test. Both those children who were above and those who were below the mean at pre-test benefited from the teaching.

Our teaching experiments also showed that pupils' initial learning of fractions is to some extent context specific: for example, the same pupils who understand that 1/3 and 2/6 are equivalent fractions when discussing the division of pizzas or chocolates may not understand that a mixture of one glass of orange concentrate with two glasses of water has the same taste as another where two glasses of orange concentrate and four glasses of water were used. This latter type of situation, which involves intensive quantities, is seldom discussed in the classroom (for more information on intensive quantities, see TLRP Research Briefing Number 10).

Major implications

Our surveys of pupils' understanding of fractions confirmed the results of others carried out approximately 20 years ago. Pupils have considerable difficulty with the conceptual aspects of rational numbers. This challenge must be confronted by schools because fractions are important in everyday life and in the world of work and are also essential when pupils continue to study mathematics in secondary school.

Both the surveys and detailed analyses of pupils' reasoning showed that primary school pupils have some insights about fractions that could be used in teaching when they solve division problems. They understand the relative nature of fractions: if one child gets half of a big cake and the other gets half of a small one, they do not receive the same amount. They also realise, for example, that you can share something by cutting it in different ways: this makes it 'different fractions but not different amounts'. Finally, they understand the inverse relation between the denominator and the quantity: the more people there are sharing something, the less each one will get.

The intervention study showed that these insights from everyday experiences with division can be systematised in the classroom, transforming them into a solid basis for children's further learning of fractions.

Although the number of lessons used in our intervention study might seem large at first glance – five to eight lessons were necessary – this is actually a small investment when we consider that the lessons were used to set the conceptual basis for the understanding of a new way of thinking about numbers. Many more lessons are invested in creating the basis for counting and natural numbers in early years.

Division situations provide a sound starting point for pupils' understanding of the logic of rational numbers but they must not be seen as the only context in which rational numbers should be taught. Further lessons on different situations where fractions are used should be carefully planned so that pupils' concepts of fractions are extended and do not remain context specific. Revisiting concepts at different levels of difficulty is in line with the directives of the National Numeracy Strategy. But in order to accomplish this, teachers must become aware of the different situations in which fractions are used and the particular difficulties that each situation entails. Teachers presently show little awareness of some of these situations; changes in the preparation of new teachers for the teaching of fractions and in-service training are urgently needed to address this problem.

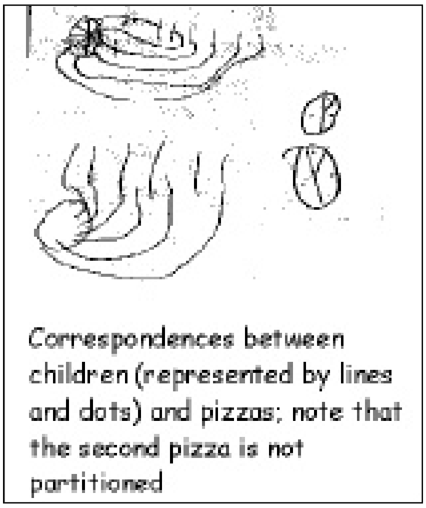

Figure 4a

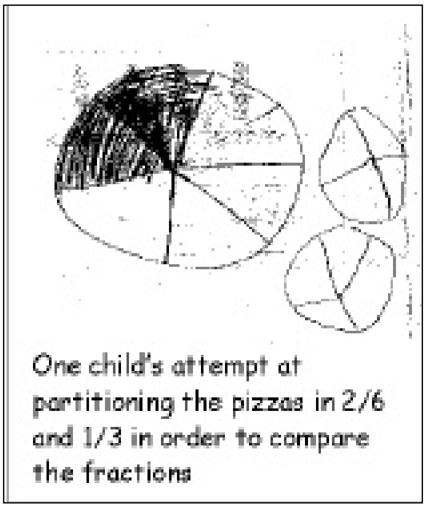

Figure 4b

This result is in line with findings in the domain of natural numbers, which also show context-specific knowledge in pupils. For example, pupils who can use subtraction to solve a change problem (Mary had 8 sweets and ate 3; how many does she have left?) may not use subtraction to solve a comparison problem where the same calculation is required (May has 8 sweets and Ali has 3; how many more sweets does May have than Ali?). But this is an important result because of the implications it has for teaching. In the domain of rational numbers, just as with natural numbers, teaching should not be restricted to a limited number of situations. It is important to extend pupils' knowledge across situations where knowledge transfer is known to cause difficulty for pupils.

Further information

Background information about children's difficulties with fractions can be obtained from:

Hart, K., Brown, M., Kerslake, D., Küchermann, D. and Ruddock, G. (1984) *Chelsea Diagnostic Mathematics Tests. Fractions 1.* Windsor (UK): NFER-Nelson.

Kerslake, D. (1986) Fr*actions: Children's Strategies and Errors: A report of the strategies and errors in Secondary Mathematics Project.* Windsor: NFER-Nelson.

Journal articles reporting the present research are currently in preparation. The project website (see below) provides further information about the results.

A conference presentation on this project is also available.

The warrant

Confidence in our conclusions can be based on the robustness of the empirical procedures, which comply in full with the highest scientific standards and were informed by the extensive experience of the project team. The sample of children in the survey covered a wide socio-economic range and all results were replicated across two cohorts of children.

The intervention problems were carefully tested by the researchers. We worked with the children in small groups and had the opportunity to record their arguments to analyse them in detail subsequently. The results were replicated in different classrooms where teachers implemented the programme in ways that they felt worked for the pupils in their classroom.

The fractions assessments were highly reliable. The inter-correlations between items were high and the test-retest correlations for the comparison groups were also high. All our conclusions are based on rigorous quantitative analyses, using inferential statistics. Reported differences between types of problems and between groups in the teaching experiment are not simply statistically significant. The effect remained stable after two months without further specific teaching.

Project website
http://www.edstud.ox.ac.uk/research/childlearning/index.html

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Teaching and Learning Research Programme

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